COMPUTATION OF HARMONIC WEAK MAASS FORMS

JAN H. BRUINIER AND FREDRIK STRÖMBERG

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Abstract. Harmonic weak Maass forms of half-integral weight are the subject of many recent works. They are closely related to Ramanujan's mock theta functions, their theta lifts give rise to Arakelov Green functions, and their coefficients are often related to central values and derivatives of Hecke L-functions. We present an algorithm to compute harmonic weak Maass forms numerically, based on the automorphy method due to Hejhal and Stark. As explicit examples we consider harmonic weak Maass forms of weight $1/2$ associated to the elliptic curves 11a1, 37a1, 37b1. We made extensive numerical computations and the data we obtained is presented in the final section of the paper. We expect that experiments based on our data will lead to a better understanding of the arithmetic properties of the Fourier coefficients.

1. Introduction

Half-integral weight modular forms play important roles in arithmetic geometry and number theory. Their coefficients serve as generating functions for various interesting number theoretic functions, such as representation numbers of quadratic forms in an odd number of variables or class numbers of imaginary quadratic fields. Moreover, employing the Shimura correspondence [Sh], Waldspurger [Wa], and Kohnen and Zagier [KZ, K] showed that the coefficients of half-integral weight cusp forms essentially are square-roots of central values of quadratic twists of modular L-functions. In analogy with these works, Katok and Sarnak [KS] used a Shimura correspondence to relate coefficients of weight $1/2$ Maass forms to sums of values and sums of line integrals of Maass cusp forms.

In more recent work Zagier discovered that the generating function for the traces of singular moduli (the CM values of the classical j-function) is a weakly holomorphic modular form of weight $3/2$ [Za1]. This result, which was generalized in various directions (see e.g. [BO2], [BF2], [DJ], [Ki]), demonstrates that also the coefficients of automorphic forms with singularities at the cusps carry interesting arithmetic information.

In a similar spirit, Ono and the first author proved that the coefficients of harmonic weak Maass forms of weight $1/2$ are related to both the values and central derivatives of quadratic twists of weight 2 modular L-functions [BruO]. Harmonic weak Maass forms are also closely related to mock modular forms and to Ramanujan's mock theta functions, which have been the subject of various recent works (see e.g. [BO1, BO3, On, Za2, Zw1, Zw2]). In view of these connections, it is desirable to develop tools for the computation of such automorphic

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forms. In the present paper we propose an approach to this problem which yields an efficient algorithm. Moreover, we compute some harmonic weak Maass forms which are related to rational elliptic curves as in [BruO].

The non-holomorphic nature of harmonic weak Maass forms prevents the use of the well developed algorithms existing for (weakly) holomorphic modular forms, such as e.g. modular symbols. The use of Poincaré series does not work well either in small weights due to the poor convergence of the infinite series which appear in the explicit formulas for the coefficients. Instead we adapt the 'automorphy method', originally developed by Hejhal for the computation of Maass cusp forms on Hecke triangle groups (see e.g. [He]), to the setting of harmonic weak Maass forms.

We now describe the content of this paper in more detail. Let $k \in \frac{1}{2}\mathbb{Z}$, and let N be a positive integer (with $4 \mid N$ if $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$). A *harmonic weak Maass form* of weight k on $\Gamma_0(N)$ is a smooth function on $\mathbb{H}$, the upper half of the complex plane, which satisfies:

(i) $f \mid_k \gamma = f$ for all $\gamma \in \Gamma_0(N)$;
(ii) $\Delta_k f = 0$, where Δ_k is the weight k hyperbolic Laplacian on $\mathbb{H}$ (see (2.3));
(iii) There is a polynomial $P_f = \sum_{n \leq 0} c^+(n) q^n \in \mathbb{C}[q^{-1}]$ such that $f(\tau) - P_f(\tau) = O(e^{-\varepsilon v})$ as $v \to \infty$ for some $\varepsilon > 0$. Analogous conditions are required at all cusps.

Throughout, for $\tau \in \mathbb{H}$, we let $\tau = u + iv$, where $u, v \in \mathbb{R}$, and we let $q := e^{2\pi i \tau}$. The polynomial P_f is called the *principal part* of f at ∞.

Such a harmonic weak Maass form f has a Fourier expansion at infinity of the form

$$f(\tau) = \sum_{n \gg -\infty} c^+(n) q^n + \sum_{n<0} c^-(n) \Gamma\left(1-k, 4\pi |n| v\right) q^n, \tag{1.1}$$

where $\Gamma(a,x)$ denotes the incomplete Gamma function. The series $\sum_{n \gg -\infty} c^+(n) q^n$ is called the *holomorphic part* of f, and its complement is called the *non-holomorphic part*. Naturally, f has similar expansions at the other cusps. There is an antilinear differential operator, taking f to the cusp form $\xi_k(f) := 2iv^k \overline{\frac{\partial f}{\partial \bar{\tau}}}$ of weight $2-k$, see (2.5). The kernel of ξ_k consists of the space of *weakly holomorphic* modular forms, those meromorphic modular forms whose poles (if any) are supported at cusps.

Every weight $2-k$ cusp form is the image under ξ_k of a weight k harmonic weak Maass form. Ramanujan's mock theta functions correspond to those forms whose images under $\xi_{1/2}$ are weight $3/2$ unary theta functions. Here we mainly consider those weight $1/2$ harmonic weak Maass forms whose images under $\xi_{1/2}$ are orthogonal to the unary theta series. According to [BruO], their coefficients are related to both the values and central derivatives of quadratic twists of weight 2 modular L-functions.

We now briefly describe this result in the special case that the level is a prime p. Let $G \in S_2(\Gamma_0(p))$ be a normalized Hecke eigenform whose Hecke L-function $L(G,s)$ satisfies an odd functional equation. That is, the completed L-function $\Lambda(G,s) = p^{s/2}(2\pi)^{-s}\Gamma(s)L(G,s)$ satisfies $\Lambda(G, 2-s) = \varepsilon_G \Lambda(G,s)$ with root number $\varepsilon_G = -1$. Therefore, the central critical value $L(G,1)$ vanishes. By Kohnen's theory of plus-spaces [K], there is a half-integral weight newform $g \in S^+_{3/2}(\Gamma_0(4p))$, unique up to a multiplicative constant, which lifts to G under the Shimura correspondence. We choose g so that its coefficients are in F_G, the

totally real number field generated by the Hecke eigenvalues of G. There exists a weight $1/2$ harmonic weak Maass form f on $\Gamma_0(4p)$ in the plus space whose principal part P_f has coefficients in F_G, and such that

$$\xi_{1/2}(f) = \|g\|^{-2} g,$$

where $\|g\|$ denotes the usual Petersson norm. For a fundamental discriminant Δ let χ_Δ be the Kronecker character for $\mathbb{Q}(\sqrt{\Delta})$, and let $L(G, \chi_\Delta, s)$ be the quadratic twist of $L(G, s)$ by χ_Δ. One can show that the root number of $L(G, \chi_\Delta, s)$ is equal to $\operatorname{sign}(\Delta) \cdot \chi_\Delta(p)\, \varepsilon_G$.

Theorem 1.1 (See [BruO]). *Assume that G, g, and f are as above, and let $c^{\pm}(n)$ denote the Fourier coefficients as in* (1.1).

(1) *If $\Delta < 0$ is a fundamental discriminant for which $\left(\frac{\Delta}{p}\right) = 1$, then*

$$L(G, \chi_\Delta, 1) = 8\pi^2 \|G\|^2 \|g\|^2 \sqrt{\frac{|\Delta|}{N}} \cdot c^-(\Delta)^2.$$

(2) *If $\Delta > 0$ is a fundamental discriminant for which $\left(\frac{\Delta}{p}\right) = 1$, then $L'(G, \chi_\Delta, 1) = 0$ if and only if $c^+(\Delta)$ is algebraic.*

Note that the harmonic weak Maass form f is uniquely determined up to the addition of a weight $1/2$ weakly holomorphic modular form with coefficients in F_G. Furthermore, the absolute values of the nonvanishing coefficients $c^+(\Delta)$ are typically asymptotic to subexponential functions in n. For these reasons, the connection between $L'(G, \chi_\Delta, 1)$ and the coefficients $c^+(\Delta)$ in Theorem 1.1(2) cannot be modified in a simple way to obtain a formula as in the first part of the Theorem. In fact, the proof of Theorem 1.1(2) is rather indirect. It relies on the Gross-Zagier formula and on transcendence results of Waldschmidt and Scholl on periods of differentials on algebraic curves.

The above result is one of the main motivations for the present paper. Our goal is to carry out numerical computations for the involved harmonic weak Maass forms. In that way we hope to find more direct connections of the coefficients $c^+(\Delta)$ to periods or L-functions. When $L'(G, \chi_\Delta, 1)$ vanishes, meaning that $c^+(\Delta)$ is algebraic (actually contained in F_G), it would be interesting to see if $c^+(\Delta)$ carries any arithmetic information related to G. In a forthcoming paper [Br2], the coefficients $c^+(n)$ will be linked to periods of certain algebraic differentials of the third kind on modular curves. It leads to a conjecture on differentials of the third kind on elliptic curves, which is based on the numerical data presented in Section 4 of the present paper.

Our computations make use of an adaption of the so-called automorphy method. The key point of this method is to view an automorphic form on a non-co-compact (but cofinite) Fuchsian group Γ as a function on the upper half-plane with certain transformation properties under the group Γ as well as convergent Fourier series expansions at all cusps. This classical point of view, in terms of functions on the upper half-plane, stands in contrast to the more algebraic point of view, in terms of Hecke modules, usually taken when computing holomorphic modular forms.

By *computing* an automorphic form ϕ in this setting we mean that to any given (small) $\epsilon > 0$ we compute a sufficient number of Fourier coefficients, each to high enough precision, so that we are able to evaluate the function ϕ at any point in the upper half-plane with an error at most ϵ.

To calculate these Fourier coefficients we truncate the Fourier series representing ϕ and view the resulting trigonometric sum as a finite Fourier series. Using the Fourier inversion theorem together with the automorphic properties of ϕ (which will additionally intertwine the Fourier series at various cusps) we are able to obtain a set of linear equations satisfied approximately by the coefficients. Cf. e.g. [He, St1, Av2]. The (surprising) effectiveness of this algorithm is closely related to the equidistribution properties of closed horocycles (cf. e.g. [He1, S]). We describe the main algorithm in detail in Section 3. The implementation of the software package is briefly described in Section 3.3.

In Section 4 we describe our computational result in three cases of particular interest. We consider the elliptic curves 11a1, 37a1, and 37b1 and their corresponding weight 2 newforms. For instance, the elliptic curve 37a1, is the curve of smallest conductor with rank 1. It corresponds to the unique weight two normalized newform G on $\Gamma_0(37)$ whose L-function has an odd functional equation. We verified the statement of Theorem 1.1 for all fundamental discriminants Δ which are squares modulo 148 in the range $0 < \Delta < 15000$. For eight of these fundamental discriminants the quantity $L'(G, \chi_\Delta, 1)$ vanishes. In all these cases we found a stronger statement then that of the Theorem 1.1 to be true, namely, that the associated coefficient $c^+(\Delta)$ was an integer. For the corresponding data see Tables 4 and 5. We conclude Section 4 by describing some analogous experiments for newforms G of weight 4, where g is of weight $5/2$ and f of weight $-1/2$.

The present paper is organized as follows. In Section 2 we recall some facts on (half integral weight) harmonic weak Maass forms. When working with arbitrary (not necessarily prime) level, it is convenient to use vector valued modular forms. In Section 2.3 we therefore recall from [BruO] the vector valued version of Theorem 1.1. In Section 3 we describe the automorphy method in the context of harmonic weak Maass forms. In Section 4 we collect our computational results. In particular, we present results for the elliptic curves 11a1, 37a1, and 37b1; cf., e.g. Tables 1, 4 and 7. More extensive tables can be obtained from the authors on request.

2. Preliminaries

In order to be able to work with newforms of arbitrary level, it is convenient to work with vector valued modular forms of half integral weight for the metaplectic extension of $\mathrm{SL}_2(\mathbb{Z})$. We describe the necessary background in this section.

2.1. A Weil representation.

Let $\mathbb{H} = \{\tau \in \mathbb{C};\ \Im(\tau) > 0\}$ be the complex upper half plane. We write $\mathrm{Mp}_2(\mathbb{R})$ for the metaplectic two-fold cover of $\mathrm{SL}_2(\mathbb{R})$, realized as the group of pairs $(M, \phi(\tau))$, where $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ and $\phi : \mathbb{H} \to \mathbb{C}$ is a holomorphic function with $\phi(\tau)^2 = c\tau + d$. The multiplication is defined by

$$(M, \phi(\tau))(M', \phi'(\tau)) = (MM', \phi(M'\tau)\phi'(\tau)).$$

We denote the inverse image of $\Gamma := \mathrm{SL}_2(\mathbb{Z})$ under the covering map by $\tilde{\Gamma} := \mathrm{Mp}_2(\mathbb{Z})$. It is well known that $\tilde{\Gamma}$ is generated by $T := \left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), 1\right)$, and $S := \left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \sqrt{\tau}\right)$.

Let N be a positive integer. There is a certain representation ρ of $\tilde{\Gamma}$ on $\mathbb{C}[\mathbb{Z}/2N\mathbb{Z}]$, the group ring of the finite cyclic group of order $2N$. For a coset $h \in \mathbb{Z}/2N\mathbb{Z}$ we denote by $\mathfrak{e}_h$ the corresponding standard basis vector of $\mathbb{C}[\mathbb{Z}/2N\mathbb{Z}]$. We write $\langle \cdot, \cdot \rangle$ for the standard scalar product (antilinear in the second entry) such that $\langle \mathfrak{e}_h, \mathfrak{e}_{h'} \rangle = \delta_{h,h'}$. In terms of the generators T and S of $\tilde{\Gamma}$, the representation ρ is given by

$$\rho(T)(\mathfrak{e}_h) = e\left(\frac{h^2}{4N}\right)\mathfrak{e}_h, \tag{2.1}$$

$$\rho(S)(\mathfrak{e}_h) = \frac{1}{\sqrt{2iN}} \sum_{h'\ (2N)} e\left(-\frac{hh'}{2N}\right)\mathfrak{e}_{h'}. \tag{2.2}$$

Here the sum runs through the elements of $\mathbb{Z}/2N\mathbb{Z}$ and we have put $e(a) = e^{2\pi i a}$. Note that ρ is the Weil representation associated to the one-dimensional positive definite lattice $K = (\mathbb{Z}, Nx^2)$ in the sense of [Bo1], [Br1], [BruO]. It is unitary with respect to the standard scalar product.

If $k \in \frac{1}{2}\mathbb{Z}$, we write $M^!_{k,\rho}$ for the space of $\mathbb{C}[\mathbb{Z}/2N\mathbb{Z}]$-valued weakly holomorphic modular forms of weight k for $\tilde{\Gamma}$ with representation ρ. The subspaces of holomorphic modular forms and cusp forms are denoted by $M_{k,\rho}$ and $S_{k,\rho}$, respectively.

2.2. **Harmonic weak Maass forms.** In this subsection we assume that $k \leq 1$. A twice continuously differentiable function $f : \mathbb{H} \to \mathbb{C}[\mathbb{Z}/2N\mathbb{Z}]$ is called a *harmonic weak Maass form* (of weight k with respect to $\tilde{\Gamma}$ and ρ) if it satisfies:

(i) $f(M\tau) = \phi(\tau)^{2k}\rho(M,\phi)f(\tau)$ for all $(M,\phi) \in \tilde{\Gamma}$;
(ii) $\Delta_k f = 0$,
(iii) there is a $\mathbb{C}[\mathbb{Z}/2N\mathbb{Z}]$-valued Fourier polynomial

$$P_f(\tau) = \sum_{h\ (2N)} \sum_{n \in \mathbb{Z}_{\leq 0}} c^+(n,h) q^{\frac{n}{4N}} \mathfrak{e}_h$$

such that $f(\tau) - P_f(\tau) = O(e^{-\varepsilon v})$ as $v \to \infty$ for some $\varepsilon > 0$.

Here we have that

$$\Delta_k := -v^2\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + ikv\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right) \tag{2.3}$$

is the usual weight k hyperbolic Laplace operator (see [BF1]). The Fourier polynomial P_f is called the *principal part* of f. We denote the vector space of these harmonic weak Maass forms by $H_{k,\rho}$ (it was called $H^+_{k,\rho}$ in [BF1]). Any weakly holomorphic modular form is a harmonic weak Maass form. The Fourier expansion of any $f \in H_{k,\rho}$ gives a unique

decomposition $f = f^+ + f^-$, where

$$f^+(\tau) = \sum_{h\ (2N)} \sum_{\substack{n\in\mathbb{Z} \\ n\gg-\infty}} c^+(n,h) q^{\frac{n}{4N}} \mathfrak{e}_h, \tag{2.4a}$$

$$f^-(\tau) = \sum_{h\ (2N)} \sum_{\substack{n\in\mathbb{Z} \\ n<0}} c^-(n,h) \Gamma\left(1-k, 4\pi\left|\frac{n}{4N}\right| v\right) q^{\frac{n}{4N}} \mathfrak{e}_h. \tag{2.4b}$$

We refer to f^+ as the *holomorphic part* and to f^- as the *non-holomorphic part* of f. Note that $c^\pm(n,h) = 0$ unless $n \equiv h^2\ (4N)$.

Recall that there is an antilinear differential operator $\xi = \xi_k : H_{k,\rho} \to S_{2-k,\bar\rho}$, defined by

$$f(\tau) \mapsto \xi(f)(\tau) := 2iv^k \overline{\frac{\partial f}{\partial \bar\tau}}. \tag{2.5}$$

Here $\bar\rho$ denotes the complex conjugate of the representation ρ, which can be identified with the dual representation. The map ξ is surjective and its kernel is the space $M^!_{k,\rho}$. There is a bilinear pairing between $M_{2-k,\bar\rho}$ and $H_{k,\rho}$ defined by the Petersson scalar product

$$\{g, f\} = \big(g,\, \xi(f)\big) := \int_{\Gamma\backslash\mathbb{H}} \langle g,\, \xi(f)\rangle v^{2-k} \frac{du\, dv}{v^2}, \tag{2.6}$$

for $g \in M_{2-k,\bar\rho}$ and $f \in H_{k,\rho}$. If g has the Fourier expansion $g = \sum_{h,n} b(n,h) q^{n/4N} \mathfrak{e}_h$, and if we denote the Fourier expansion of f as in (2.4), then by [BF1, Proposition 3.5] we have

$$\{g, f\} = \sum_{h\ (2N)} \sum_{n\le 0} c^+(n,h) b(-n,h). \tag{2.7}$$

Hence $\{g, f\}$ only depends on the principal part of f.

2.3. **The Shimura lift.** Let $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$. According to [EZ, Chapter 5], the space $M_{k,\bar\rho}$ is isomorphic to $J_{k+1/2,N}$, the space of holomorphic Jacobi forms of weight $k+1/2$ and index N. According to [Sk1] and [SZ], $M_{k,\rho}$ is isomorphic to $J^{skew}_{k+1/2,N}$, the space of skew holomorphic Jacobi forms of weight $k+1/2$ and index N. There is an extensive Hecke theory for Jacobi forms (see [EZ], [Sk1], [SZ]), which gives rise to a Hecke theory on $M_{k,\rho}$ and $M_{k,\bar\rho}$, and which is compatible with the Hecke theory on vector valued modular forms considered in [BrSt]. In particular, there is an Atkin-Lehner theory for these spaces.

The subspace $S^{new}_{k,\rho}$ of newforms of $S_{k,\rho}$ is isomorphic as a module over the Hecke algebra to the space of newforms $S^{new,+}_{2k-1}(N)$ of weight $2k-1$ for $\Gamma_0(N)$ on which the Fricke involution acts by multiplication with $(-1)^{k-1/2}$. The isomorphism is given by the Shimura correspondence. Similarly, the subspace $S^{new}_{k,\bar\rho}$ of newforms of $S_{k,\bar\rho}$ is isomorphic as a module over the Hecke algebra to the space of newforms $S^{new,-}_{2k-1}(N)$ of weight $2k-1$ for $\Gamma_0(N)$ on which the Fricke involution acts by multiplication with $(-1)^{k+1/2}$ (see [SZ], [GKZ], [Sk1]). Observe that the Hecke L-series of any $G \in S^{new,\pm}_{2k-1}(N)$ satisfies a functional equation under $s \mapsto 2k-1-s$ with root number $\varepsilon_G = \pm 1$.

We now state the vector valued version of Theorem 1.1. Let $G \in S^{new}_2(N)$ be a normalized newform (in particular a common eigenform of all Hecke operators) of weight 2

and write F_G for the number field generated by the eigenvalues of G. If $\varepsilon_G = -1$ we put $\rho' = \rho$, and if $\varepsilon_G = +1$ we put $\rho' = \bar{\rho}$. There is a newform $g \in S^{new}_{3/2,\bar{\rho}'}$ mapping to G under the Shimura correspondence. It is well known that we may normalize g such that all its coefficients are contained in F_G. According to [BruO, Lemma 7.3], there is a harmonic weak Maass form $f \in H_{1/2,\rho'}$ whose principal part has coefficients in F_G with the property that

$$\xi_{1/2}(f) = \|g\|^{-2} g.$$

This form is unique up to addition of a weakly holomorphic form in $M^!_{1/2,\rho'}$ whose principal part has coefficients in F_G.

In practice, the principal part of such an f can be computed as follows: We may complete the weight 3/2 form g to an orthogonal basis $g, g_2, \ldots, g_d$ of $S_{3/2,\bar{\rho}'}$ consisting of cusp forms with Fourier coefficients in F_G. Let $f \in H_{1/2,\rho'}$ such that

$$\{f, g\} = 1, \quad \text{and } \{f, g_i\} = 0 \text{ for } i = 2, \ldots d. \tag{2.8}$$

Then f has the required properties. In view of (2.7) the conditions of (2.8) translate into an inhomogeneous system of linear equations for the principal part of f.

Theorem 2.1. *Let $G \in S_2^{new}(N)$ be a normalized newform. Let $g \in S^{new}_{3/2,\bar{\rho}'}$, and $f \in H_{1/2,\rho'}$ be as above. Denote the Fourier coefficients of f by $c^{\pm}(n,h)$ for $n \in \mathbb{Z}$ and $h \in \mathbb{Z}/2N\mathbb{Z}$. Then the following are true:*

(1) *If $\Delta \neq 1$ is a fundamental discriminant and $r \in \mathbb{Z}$ such that $\Delta \equiv r^2 \pmod{4N}$ and $\varepsilon_G \Delta > 0$, then*

$$L(G, \chi_\Delta, 1) = 8\pi^2 \|G\|^2 \|g\|^2 \sqrt{\frac{|\Delta|}{N}} \cdot c^-(\Delta)^2.$$

(2) *If $\Delta \neq 1$ is a fundamental discriminant and $r \in \mathbb{Z}$ such that $\Delta \equiv r^2 \pmod{4N}$ and $\varepsilon_G \Delta < 0$, then*

$$L'(G, \chi_\Delta, 1) = 0 \iff c^+(-\varepsilon_G \Delta, r) \in \bar{\mathbb{Q}} \iff c^+(-\varepsilon_G \Delta, r) \in F_G.$$

When $S_{1/2,\rho'} = \{0\}$ the above result also holds for $\Delta = 1$, see also [BruO, Remark 18]. This is for instance the case when N is a prime. If N is a prime and $\varepsilon_G = -1$, then the space $H_{1/2,\rho'}$ can be identified with a space of scalar valued modular forms satisfying a Kohnen plus space condition. In that way one obtains Theorem 1.1 stated in the introduction.

3. Computational aspects

3.1. **The automorphy method for vector valued weak Maass forms.** To compute the Fourier coefficients of the harmonic weak Maass forms we use the so-called automorphy method, sometimes called "Hejhal's method". This is a general method which has been used to successfully compute various kinds of automorphic functions and forms on $\mathrm{GL}_2(\mathbb{R})$. It was originally developed by Hejhal in order to compute Maass cusp forms for the modular group and other Hecke triangle groups (cf. e.g. [He]). The method was later generalized by the second author in [St1] to computations of Maass waveforms with non-trivial multiplier systems and arbitrary real weights, as well as to general subgroups of the modular group

(see also [St3]). Another generalization to automorphic forms with singularities (Eisenstein series, Poincaré series and Green's functions) was made by Avelin [Av1, Av2].

We will detail the adaptation of the algorithm to the case of vector-valued harmonic weak Maass forms for the Weil representation.

For simplicity consider the representation ρ (the case of $\overline{\rho}$ is analogous) and $k \in \mathbb{Z} + \frac{1}{2}$. Furthermore, in order to avoid questions of uniqueness we assume that either $k < 0$ or that $k = \frac{1}{2}$ and that N is prime. In these cases, a harmonic weak Maass form is uniquely determined by its principal part. For computational purposes it is not feasible to use the definition of ρ in terms of the action on the generators of the metaplectic group. We instead use formulas from [St1] to evaluate ρ on the fixed (canonical) representative of $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, i.e. $\rho(M) := \rho(M, j_M(\tau))$ where $j_M(\tau) = \sqrt{c\tau + d}$ is defined by the principal branch of the argument.

3.1.1. *The algorithm – phase 1.* Let $f \in H_{k,\rho}$ with a given (fixed) principal part $P_f(\tau) = \sum_h P_{f,h}(\tau)\,\mathfrak{e}_h$ where $P_{f,h}(\tau) = \sum_{n=-K}^{0} a(n,h)\,q^{\frac{n}{4N}}$ (for some finite $K \geq 0$) and write $f = f^+ + f^-$ (as in 2.3a and 2.3b) with $f^+ = \sum_{h(2N)} f_h^+ \mathfrak{e}_h$ and $f^- = \sum_{h(2N)} f_h^- \mathfrak{e}_h$ where

$$f_h^+(\tau) = \sum_{n=-K}^{0} a(n,h)\,q^{\frac{n}{4N}} + \sum_{n>0} c^+(n,h)\,q^{\frac{n}{4N}} \quad \text{and}$$
$$f_h^-(\tau) = \sum_{n<0} c^-(n,h)\,\Gamma\left(1-k, 4\pi\left|\frac{n}{4N}\right| v\right) q^{\frac{n}{4N}}$$

for $\tau = u + iv \in \mathbb{H}$. Our goal is to obtain numerical approximations to the coefficients $c^{\pm}(n,h)$. To formulate our algorithm we prefer to separate the u- and the v-dependence in f and therefore introduce the function W defined by $W(v) = e^{-2\pi v}$ if $v > 0$ and $W(v) = e^{-2\pi v}\Gamma(1-k, 4\pi|v|)$ if $v < 0$. We also set $c(n,h) = c^+(n,h)$ for $n > 0$ and $c^-(n,h)$ for $n < 0$ and write $e_{4N}(u) = e^{\frac{2\pi i u}{4N}}$. With this notation

$$f_h(\tau) = \sum_{n=-K}^{0} a(n,h)\,q^{\frac{n}{4N}} + \sum_{n\neq 0} c(n,h)\,W\left(\frac{nv}{4N}\right) e_{4N}(nu).$$

By standard inequalities for the incomplete gamma function one can show that

$$|W(v)| < c_k\, e^{-2\pi|v|} \begin{cases} 1, & v > 0, \\ (4\pi|v|)^{-k}, & v < 0, \end{cases}$$

where c_k is an explicit constant only depending on k. To be able to determine a truncation point of the Fourier series above we also need bounds of the coefficients $c(n,h)$. Using [BruFu, Lemma 3.4] it follows that there exists an explicit constant $C > 0$ such that

$$c(n,h) = O\left(\exp\left(4\pi C\sqrt{n}\right)\right), \quad n \to +\infty,$$
$$c(n,h) = O(|n|^{\frac{k}{2}}), \quad n \to -\infty.$$

For $k < 0$ we are able to make the implied constants explicit using non-holomorphic Poincaré series as in e.g. [Br1] or [He2]. For $k = \frac{1}{2}$ we rely on numerical a posteriori tests to assure ourselves that the truncation point was choosen correctly. See e.g. Section 3.2.

Let $\epsilon > 0$ and fix $Y < Y_0 = \frac{\sqrt{3}}{2}$. By the estimates above we can find an $M_0 = M(Y, \epsilon)$ such that the function $\hat{f} = \sum_{h\,(2N)} \hat{f}_h \mathfrak{e}_h$ given by the truncated Fourier series

$$\hat{f}_h(\tau) = P_{f,h}(\tau) + \sum_{0<|n|\le M_0} c(n,h)\, W\left(\frac{nv}{4N}\right) e_{4N}(nu)$$

satisfies

$$\left\| \hat{f}(\tau) - f(\tau) \right\|^2 < \epsilon$$

for any $\tau \in \mathcal{H}_Y = \{\tau \in \mathcal{H} \mid \Im\tau \ge Y\}$. Here $\|z\|^2 = \sum_{h=1}^{2N} |z_h|^2$ for $z \in \mathbb{C}^{2N}$. Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and set $z = x + iy = A\tau$. Then $y = \Im A\tau = \frac{v}{|c\tau+d|^2} \le \frac{v}{c^2v^2} \le \frac{1}{v}$ and hence $|j_A(\tau)|^4 = |c\tau + d|^2 = \frac{v}{y} \le \frac{1}{y^2}$. Using the fact that ρ is unitary it is now easy to see that if $\tau, A\tau \in \mathcal{H}_Y$ then

$$\left\| \hat{f}(A\tau) - j_A(\tau)^{2k} \rho(A) \hat{f}(\tau) \right\|^2 < \epsilon\left(1 + Y^{-2k}\right) < 2\epsilon \cdot Y^{-2k}. \tag{3.1}$$

Consider now a horocycle at height Y and a set of $2Q$ (with $Q > M_0$) equally spaced points

$$z_m = x_m + iY, \quad x_m = \frac{1-2m}{4Q}, \quad 1 - Q \le m \le Q.$$

If we view the series $\hat{f}_h$ as a finite Fourier series we can invert it over this horocycle and it is easy to see that if n is an integer with $0 < |n| \le M_0$ and $n \equiv h^2\,(4N)$ then

$$\frac{1}{2Q} \sum_{m=1-Q}^{Q} \hat{f}_h(z_m)\, e_{4N}(-nx_m) = W\left(\frac{n}{4N}Y\right) c(n,h) + a(n,h)\; e^{-\frac{2\pi n}{4N}Y}. \tag{3.2}$$

One can also interpret the left-hand side as a Riemann-sum approximation to the integral

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} f_h(z)\, e_{4N}(-nx)\, dx.$$

Let $z_m^* = x_m^* + iy_m^* = T_m^{-1} z_m$ ($T_m \in \mathrm{PSL}_2(\mathbb{Z})$) denote the pull-back of z_m into the standard (closed) fundamental domain of $\mathrm{PSL}_2(\mathbb{Z})$, $\mathcal{F} = \left\{ z = x + iy \mid |x| \le \frac{1}{2},\, |z| \ge 1 \right\}$. Using (3.1) we obtain

$$\hat{f}_h(z_m) = j_{T_m}(z_m^*) \sum_{h'\,(2N)} \rho_{hh'}(T_m)\, \hat{f}_{h'}(z_m^*) + [\![2\epsilon Y^{-2k}]\!],$$

where $\rho_{hh'}(T_m)$ is the (h,h')-element of the matrix $\rho(T_m)$, and we use $[\![2\epsilon Y^{-2k}]\!]$ to denote a quantity bounded in absolute value by $2\epsilon Y^{-2k}$. Inserting this into (3.2) we see that the

left-hand side can be written as

$$\begin{aligned}
&\frac{1}{2Q}\sum_{m=1-Q}^{Q} j_{T_m}\left(z_m^*\right)\sum_{h'\,(2N)}\rho_{hh'}\left(T_m\right)\left[\sum_{l=-K}^{0} a(l,h')\exp\left(-\frac{2\pi l}{4N}y_m^*\right)e_{4N}\left(lx_m^*\right)\right.\\
&\qquad\qquad\left.+\sum_{0<|l|\le M_0} c(l,h')W\left(\frac{l}{4N}y_m^*\right)e_{4N}\left(lx_m^*\right)\right]e_{4N}(-nx_m)\\
&\qquad\qquad=\sum_{h'\,(2N)}\sum_{0<|l|\le M_0} c\left(l,h'\right)\widetilde{V}_{nl}^{hh'}+\widetilde{W}_n^h+[\![2\epsilon Y^{-2k}]\!],
\end{aligned}\tag{3.3}$$

where

$$\widetilde{V}_{nl}^{hh'}=\frac{1}{2Q}\sum_{m=1-Q}^{Q} j_{T_m}\left(z_m^*\right)\rho_{hh'}\left(T_m\right)W\left(\frac{l}{4N}y_m^*\right)e_{4N}(lx_m^*-nx_m)\quad\text{and}$$

$$\widetilde{W}_n^h=\frac{1}{2Q}\sum_{h'\,(2N)}\sum_{l=-K}^{0} a\left(l,h'\right)\sum_{m=1-Q}^{Q} j_{T_m}\left(z_m^*\right)\rho_{hh'}\left(T_m\right)\exp\left(-\frac{2\pi l}{4N}y_m^*\right)e_{4N}\left(lx_m^*-nx_m\right).$$

We thus have an inhomogeneous system of linear equations which is (approximately) satisfied by the coefficients $c\,(n,h)$. Let $\mathcal{D}=\{(n,h)\mid 0<|n|\le M_0,\ 0\le h<2N\}$ (with a fixed ordering) and note that $|\mathcal{D}|=4M_0N$. If we set $\vec{D}=\left(d\,(n,h)\right)_{(n,h)\in\mathcal{D}}$,

$$\begin{aligned}
V&=V\left(Y\right)=\left(V_{nl}^{hh'}\right)_{(h,n),(h',l)\in\mathcal{D}}, & V_{nl}^{hh'}&=\widetilde{V}_{nl}^{hh'}-\delta_{nl}\delta_{hh'}W\left(\frac{n}{4N}Y\right)\quad\text{and}\\
\vec{W}&=\vec{W}\left(Y\right)=\left(W_n^h\right)_{(h,n)\in\mathcal{D}}, & W_n^h&=\widetilde{W}_n^h-a\left(n,h\right)e^{-\frac{2\pi n}{4N}Y},
\end{aligned}$$

we can write this linear system as $|\mathcal{D}|$ linear equations in $|\mathcal{D}|$ variables:

$$V\vec{D}+\vec{W}=\vec{0}.\tag{3.4}$$

In practice it turns out that the the matrix V is non-singular as soon as the subspace of $H_{k,\rho}$ consisting of functions with a given singular part is one-dimensional. In these cases we can immediately obtain the solution as

$$\vec{D}=-V^{-1}\vec{W},$$

and since we know that the vector of the "true" coefficients, $\vec{C}=\left(c\,(n,h)\right)_{(n,h)\in\mathcal{D}}$, satisfies

$$\left\|V\vec{C}+\vec{W}\right\|_\infty\le 2\epsilon Y^{-2k},$$

we see that

$$\left\|\vec{C}-\vec{D}\right\|_\infty=\left\|\vec{C}+V^{-1}\vec{W}\right\|_\infty\le\left\|V^{-1}\right\|_\infty\cdot\left\|V\vec{C}+\vec{W}\right\|_\infty\le 2\epsilon Y^{-2k}\left\|V^{-1}\right\|_\infty.$$

To obtain a theoretical error estimate we would thus need to estimate $\|V^{-1}\|_\infty$ from below. Unfortunately this does not seem to be possible from the formulas above and we have to use numerical methods to estimate this norm. Hence, to obtain the Fourier coefficients up

to a (proven) desired precision we might have to go back and decrease the original ϵ or increase either of M_0 or Q.

At this point one should also remark that the error bound $\|V^{-1}\|_\infty$ is in general much worse than the actual apparent error, as verified by studying coefficients known to be integers. The reason for this is that the sums $\widetilde{V}_{nl}^{hh'}$ exhibit massive cancellation and are therefore overpowered by the terms $W\left(\frac{n}{4N}Y\right)$ on the diagonal.

3.1.2. *The algorithm – phase 2.* Returning to (3.3) and solving for $c(n,h)$ we see that

$$c(n,h) = W\left(\frac{n}{4N}Y\right)^{-1}\left[\sum_{h'\,(2N)}\sum_{|l|\le M_0} c(l,h')\,\widetilde{V}_{nl}^{hh'} + W_n^h + [\![2\epsilon Y^{-2k}]\!]\right] \tag{3.5}$$

for *any* n, i.e. also when $|n| > M_0$, provided that $Q > M(Y)$. If we first choose Y such that $W\left(\frac{n}{4N}Y\right)$ is not too small then we can in fact use this equation to compute $c(n,h)$ with an error of size $\epsilon\, W\left(\frac{n}{4N}Y\right)^{-1}$. In this manner, we may produce long stretches of coefficients (before we need to decrease Y again) at arbitrary intervals $N_A \le n \le N_B$ without the need of computing intermediate coefficients above the initial set up to $n = M_0$.

Remark 1. The exact same algorithm, with the non-holomorphic parts set to zero, also lets one compute holomorphic vector-valued modular forms for the Weil representation. This has been exploited by the second author, in verifying computations of holomorphic Poincaré series in [RSS].

3.2. **Heuristic error estimates.** For $k < 0$ all implied constants and therefore all error estimates can be made explicit. In the remaining case which interests us, $k = \frac{1}{2}$, the known bounds for the twisted Kloosterman sums are not enough to prove the necessary explicit bounds for the Fourier coefficients of the associated Poincaré series. We are therefore not able to give effective theoretical error estimates in this case. However, this is not a serious problem since there are a number of tests we may perform on the resulting coefficients to assure ourselves of their accuracy. We list a few tests which we have used.

- First of all, one can simply use two different values of Y and verify that the resulting vectors $\vec{D} = \vec{D}(Y)$ are independent of Y.

This test is completely general and can be used for all instances where the algorithm can be applied. Suppose now that we have a harmonic weak Maass form $f \in H_{k,\rho}$ of half-integral weight k such that $\xi_k(f) = \|g\|^{-2}g$, with $g \in S_{2-k,\bar{\rho}}$. We then know the following.

- The coefficients $\sqrt{|\Delta|}c^-(-\varepsilon_G\cdot\Delta)$ are proportional to the coefficients $b(\varepsilon_G\cdot\Delta)$ of g (cf. e.g. [BruO, p. 3]).

If additionally the Shimura lift of g is a newform $G \in S_{3-2k}^{new}(\Gamma_0(N))$ then we can predict that certain coefficients $c^+(\Delta)$ are algebraic (cf. e.g. [BruO, Sect. 7]) and if we are able to identify these coefficients as algebraic numbers to a certain precision this can be used as another measure of the accuracy.

3.3. **Implementation.** The first implementation of the above described algorithm was made in Fortran 90, using the package ARPREC [AR] for arbitrary (fixed) precision arithmetic. The second and more recent implementation was done in Sage [SA], using the included package mpmath for arbitrary (fixed) precision arithmetic. The algorithms are currently under development but can be obtained on request from the authors. The final format we intend for these algorithms are standard classes for computing with vector and scalar-valued harmonic weak Maass forms in Sage or Purple Sage.

4. Results

4.1. **Harmonic Maass forms corresponding to elliptic curves.** In this section we present the numerical results we have obtained for harmonic weak Maass forms corresponding to weight two holomorphic forms associated to elliptic curves. We have concentrated on three particular examples. In Cremona's notation, these correspond to the curve $11a1$ of level 11 and the two curves $37a1$ and $37b1$ of level 37.

Recall that if the holomorphic weight 2 newform G of level N has Atkin-Lehner eigenvalue ± 1 then the L-function $L(G,s)$ has root number $\varepsilon_G = \mp 1$. Furthermore, since the root number of the twisted L-function $L(G,\chi_\Delta,s)$ is $\operatorname{sign}(\Delta)\chi_\Delta(N)\varepsilon_G$ and we always consider fundamental discriminants for which $\chi_\Delta(N) = 1$ we see that the central value $L(G,\chi_\Delta,1)$ vanishes if $\operatorname{sign}(\Delta)\varepsilon_G = -1$, i.e., if $L(G,s)$ has an even functional equation we consider $\Delta < 0$ and otherwise $\Delta > 0$.

For each of these examples we computed a large set of central derivatives of the twisted L-functions with the appropriate Δ using Sage and the standard algorithms there which were developed by Dokchitser. We then fixed a harmonic weak Maass form with non-zero principal part P_f such that $\xi_{\frac{3}{2}}(f)$ maps to G under the Shimura lift. In all cases we took a Poincaré series $P_{-\Delta}$ having principal part $q^{-\frac{\Delta}{4N}}$ and computed an initial set of Fourier coefficients for this function using the methods described in the previous section. We then used the second phase of the algorithm and computed more Fourier coefficients.

Note that for the results in this section, all initial "phase 1" computations were all performed using the new Sage package and all further, "phase 2", computations were done in Fortran 90.

We would like to give a flavour of the cpu-times involved. The initial computations, using our Sage code, took in all cases approximately 2 hours on a 2.66GHz Xeon processor. On the same processor, the cpu time for a single stretch of phase 2 calculations range between less than an hour for the smallest discriminant up to several days for the largest discriminant.

As a measure of the accuracy of our computations one can consider the difference between the coefficients in Tables 2, 5 and 8 and the nearest integer (the third column). To further support the correctness we also list, in Tables 3, 6 and 4.4, normalized coefficients of the non-holomorphic parts, i.e. $\sqrt{|\Delta|}c^-(\Delta)/\sqrt{|\Delta_0|}c^-(\Delta_0)$ by some fixed non-zero coefficient of index Δ_0.

4.1.1. *11a1.* Here the unique newform of weight two and level 11 is given by

$$G = \eta(\tau)^2\eta(11\tau)^2 = q - 2q^2 - q^3 + 2q^4 + q^5 + \cdots \in S_2^{new}(\Gamma_0(11))$$

and the corresponding L-function $L(G,s)$ has an even functional equation. Using Sage we computed all values of $L'(G,\chi_\Delta,1)$ for fundamental discriminants $\Delta < 0$ such that $\left(\frac{\Delta}{11}\right) = 1$ and $|\Delta| \leq 19703$. This set consists of 2749 fundamental discriminants and amongst these we found 14 discriminants for which $L'(G,\chi_\Delta,1)$ vanished up to the numerical precision (see Table 2).

As a representative for the harmonic weak Maass form in the space $H_{1/2,\bar{\rho}}$ corresponding to G, we choose the Poincaré series P_{-5} with the principal part $q^{-\frac{5}{44}}(\mathfrak{e}_7 - \mathfrak{e}_{-7})$. To compute the Fourier coefficients of P_{-5} we used the method described in the previous section with an initial $\varepsilon = 10^{-40}$ and $Y = 0.5$, which gave us a truncation point of $M_0 = 42$, corresponding to Δ between -1847 and 1885. For a short selection of computed values of $c^+(\Delta)$ see Table 1 and for a table of coefficients corresponding to all vanishing $L'(G,\chi_\Delta,1)$ see Table 2. The first few normalized "negative" coefficients are displayed in Table 3. These values should be compared to the list in [Sk2, p. 505].

4.1.2. *37a1.* Consider the newform of weight two and level 37 which has an odd functional equation. The q-expansion is given by

$$G = q - 2q^2 - 3q^3 + 2q^4 - 2q^5 + 6q^6 - q^7 + 6q^9 + 4q^{10} - 5q^{11} + \cdots \in S_2^{new}(\Gamma_0(37)).$$

Using Sage we computed all values of $L'(G,\chi_\Delta,1)$ for fundamental discriminants $\Delta > 0$ such that $\left(\frac{\Delta}{37}\right) = 1$ and $|\Delta| \leq 15000$. This set consists of 2217 fundamental discriminants and amongst these we found 8 discriminants for which $L'(G,\chi_\Delta,1)$ vanished up to the numerical precision (see Table 5). For the corresponding harmonic weak Maass form in $H_{1/2,\rho}$ we took P_{-3}, which has a principal part $q^{-\frac{3}{148}}(\mathfrak{e}_{21} + \mathfrak{e}_{21})$. The initial computation was done in Sage, using $\varepsilon = 1 \cdot 10^{-35}$, which gave a value of $M_0 = 30$, corresponding to discriminants in the range $-4440 \leq \Delta \leq 4585$. For examples of the coefficients $c^+(\Delta)$ see Tables 4 and 5. The first few normalized "negative" coefficients are displayed in Table 6.

4.1.3. *37b1.* In this case we consider the newform of weight two and level 37 which has an even functional equation. The q-expansion is given by

$$G = q + q^3 - 2q^4 - q^7 - 2q^9 + 3q^{11} + \cdots \in S_2^{new}(\Gamma_0(37)).$$

Using Sage we computed all values of $L'(G,\chi_\Delta,1)$ for fundamental discriminants $\Delta < 0$ such that $\left(\frac{\Delta}{37}\right) = 1$ and $|\Delta| \leq 12000$. This set consists of 1631 fundamental discriminants and amongst these we found 15 discriminants for which $L'(G,\chi_\Delta,1)$ vanished up to the numerical precision (see Table 8). For the corresponding harmonic weak Maass form in $H_{1/2,\bar{\rho}}$ we took P_{-12}, which has a principal part $q^{-\frac{12}{148}}(\mathfrak{e}_{30} - \mathfrak{e}_{30})$. The initial computation was done in Sage, using $\varepsilon = 1 \cdot 10^{-30}$, which gave a value of $M_0 = 33$, corresponding to discriminants in the range $-4883 \leq \Delta \leq 5029$. For examples of the coefficients $c^+(\Delta)$ see Tables 7 and 8. The first few normalized "negative" coefficients are displayed in Table 4.4.

4.2. **Conclusions of the numerical experiments for weight two.** In each of the examples of weight two newforms that we studied we saw agreement with the theorem, i.e. the coefficients $c^+(\Delta)$ (for fundamental discriminants with the appropriate property) were only algebraic when the corresponding central derivative $L'(G, \chi_\Delta, 1)$ vanished. Furthermore, we observed that in the cases we considered, the algebraic coefficients $c^+(\Delta)$ were in fact even rational *integers*.

4.3. **Further computations.** To investigate whether a result analogous to Theorem 1.1 also holds for newforms of weight 4, we computed $L'(2, G, \chi_\Delta)$ for all newforms G of weight 4 on $\Gamma_0(N)$ whith $5 \leq N \leq 150$ and fundamental discriminants Δ with $|\Delta| \leq 300$ and the property that the twisted L-function $L(s, G, \chi_\Delta)$ has an odd functional equation. For $5 \leq N \leq 10$ we additionally computed these values for fundamental discriminants Δ with $|\Delta| \leq 5000$. Amongst all these values we did not find a single example of a vanishing derivative. Even though we did not get any positive case where we could test the theorem we still wanted to make sure that there was no easily accesible counter example.

We therefore computed the Fourier coefficients, up to 40 digits precision, of the associated weight $-\frac{1}{2}$ harmonic Maass form corresponding to all weight 4 newforms defined over $\mathbb{Q}$ for N up to 100. To test the accuracy (and making sure that the implementation was correct) we did not only rely on the provable error bounds, but also checked algebraicity of certain coefficients corresponding to non-fundamental discriminants. These coefficients were indeed all found to be integers or rational with fairly small denominators. In contrast to this, the Fourier coefficients corresponding to fundamental discriminants were found not to be similarly "simple" rational numbers.

The L-value computations were performed in Sage [SA], using the included version of Rubinstein's lcalc library [L].

4.4. **Tables.**

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TABLE 1. $E = 11a1$, $P_{-5} \in H_{\frac{1}{2},\bar{\rho}}$

Δ	$c^+(\Delta)$	$L'(G, \chi_\Delta, 1)$
-7	$2.8463370190285980186651576519711751393948073861988 \cdot 10^{00}$	$1.22556687406888 \cdot 10^{00}$
-8	$2.5138482002575729165892711124435774460030012341762 \cdot 10^{00}$	$1.88791354720393 \cdot 10^{00}$
-19	$4.8192428148963255861870924437043119000673519519042 \cdot 10^{00}$	$7.51391655667451 \cdot 10^{00}$
-24	$5.1018494088339703187507177747597327598232414476402 \cdot 10^{00}$	$4.02744559024300 \cdot 10^{00}$
-35	$4.7515892636101723769649079639675162004017245362399 \cdot 10^{00}$	$7.64786334637073 \cdot 10^{00}$
-39	$-1.6466690697010481166272091028219327677356442914804 \cdot 10^{01}$	$2.97721567216550 \cdot 10^{00}$
-40	$1.1470941388138074683747768314723689292860539962900 \cdot 10^{01}$	$5.58789208952436 \cdot 10^{00}$
-43	$-1.7622439638503327722737780360046423237568367805048 \cdot 10^{01}$	$1.18814465355690 \cdot 10^{01}$
-51	$2.0736222999878741718629718432682995552582880786065 \cdot 10^{01}$	$1.30416363302768 \cdot 10^{01}$
-52	$1.5723528683914990387103216700146317562411438497615 \cdot 10^{01}$	$5.14853759817659 \cdot 10^{00}$
-68	$9.6889673322938493992006043404127469979247370067926 \cdot 10^{00}$	$3.80344864881298 \cdot 10^{00}$
-79	$1.7557351755436160739388564340027760291317089229254 \cdot 10^{01}$	$4.75620653690677 \cdot 10^{00}$
-83	$-7.1767664383427675609861242907417950544611683162859 \cdot 10^{01}$	$6.43843846621214 \cdot 10^{00}$
-84	$6.1666200626587315159968126799603650525586539365601 \cdot 10^{01}$	$6.53746327159376 \cdot 10^{00}$
-87	$-7.7230036424433334541697484050338439023979647483280 \cdot 10^{01}$	$2.35584785481347 \cdot 10^{00}$
-95	$7.8467572084064151556661839046504144426199227897994 \cdot 10^{01}$	$3.03660486030085 \cdot 10^{00}$
-811	$3.0046247983067285336553431175489765847382042907105 \cdot 10^{06}$	$1.25949136911120 \cdot 10^{01}$
-820	$-6.0493754250387304262091147332158578046510749019315 \cdot 10^{06}$	$1.19119437485937 \cdot 10^{01}$
-824	$-5.7985199999999999999999999999999999999999999999999 \cdot 10^{06}$	$-6.6 \cdot 10^{-24}$
-827	$1.8535489407871222859528059423736067736521222528554 \cdot 10^{06}$	$1.60961273159300 \cdot 10^{01}$
-831	$-6.7911392225835416131083026699411310608420151994986 \cdot 10^{06}$	$3.36744068632019 \cdot 10^{00}$
-996	$-3.5516294505685820400211045047063422129168082892941 \cdot 10^{07}$	$1.15828152096335 \cdot 10^{01}$
-1003	$1.0811934742079303073802766406181476437608668928409 \cdot 10^{07}$	$2.53076681967579 \cdot 10^{01}$
-1007	$-3.9469248000000000000000000000000000000000000000000 \cdot 10^{07}$	$-1.1 \cdot 10^{-22}$
-1011	$-3.7685140824429636488934775010060106547341275101054 \cdot 10^{07}$	$1.84592490209627 \cdot 10^{01}$
-1019	$3.3790315957549749442218769650593817997888628818338 \cdot 10^{07}$	$1.68145450009782 \cdot 10^{01}$

Table 2. $E = 11a1$, $P_{-5} \in H_{\frac{1}{2},\bar{\rho}}$

Δ	$c^+(\Delta)$	$\lvert c^+(\Delta) - [c^+(\Delta)]\rvert$
-824	-5798520	$3.0 \cdot 10^{-76}$
-1799	-2708450784	$2.7 \cdot 10^{-46}$
-4399	-68135748249936640	$2.3 \cdot 10^{-21}$
-8483	214445760716391388216704	$9.1 \cdot 10^{-28}$
-11567	$-1241226714909991920509289456$	$1.6 \cdot 10^{-25}$
-14791	$668501792910210190127098320995 20$	$3.1 \cdot 10^{-30}$
-15487	$-478732239405182448762415030881280$	$5.6 \cdot 10^{-30}$
-15659	$-804489814454597618648064770159415$	$6.8 \cdot 10^{-30}$
-15839	$-1162122495004344641799524116135680$	$7.0 \cdot 10^{-30}$
-16463	$4542575922533728228643934862230144$	$1.5 \cdot 10^{-30}$
-17023	$-23302350713109514450879400185948800$	$2.0 \cdot 10^{-29}$
-17927	$110133238181959291703634158808374784$	$1.2 \cdot 10^{-29}$
-18543	$464726791864282489334104058164482624$	$1.9 \cdot 10^{-29}$

Table 3. $E = 11a1$, $P_{-5} \in H_{\frac{1}{2},\bar{\rho}}$. Coefficients are scaled by $c^-(1)$.

Δ	$\sqrt{\Delta}\, c^-(\Delta)$	$\lvert c^-(\Delta) - [c^-(\Delta)]\rvert$
4	-3	$2.0 \cdot 10^{-100}$
5	5	$2.1 \cdot 10^{-99}$
9	-2	$1.7 \cdot 10^{-100}$
12	5	$8.0 \cdot 10^{-100}$
16	4	$1.5 \cdot 10^{-99}$
20	5	$1.1 \cdot 10^{-100}$
25	0	$1.0 \cdot 10^{-100}$
36	6	$1.0 \cdot 10^{-99}$
37	5	$4.2 \cdot 10^{-99}$
45	0	$6.4 \cdot 10^{-99}$

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TABLE 4. $E = 37a1$, $P_{-3} \in H_{\frac{1}{2},\rho}$

Δ	$c^+(\Delta)$	$L'(G,\chi_\Delta,1)$
1	$-2.8176178498959956879756075537515493438922975370716 \cdot 10^{-01}$	$3.05999773834052 \cdot 10^{-01}$
12	$-4.8852723826201225228227029607337071669095284814788 \cdot 10^{-01}$	$4.29861479867736 \cdot 10^{00}$
21	$-1.7273925723265275652082007397068992218426924398791 \cdot 10^{-01}$	$9.00238680032537 \cdot 10^{00}$
28	$6.7819399530394779828450578400669420938246859928076 \cdot 10^{-01}$	$4.32726024966011 \cdot 10^{00}$
33	$5.6630232015906998168220545669245622604190884430064 \cdot 10^{-01}$	$3.62195679113882 \cdot 10^{00}$
37	$-9.1326561374611652958506448407204050631184401026129 \cdot 10^{-01}$	$3.47328771649229 \cdot 10^{00}$
40	$4.0098509269543637915254766073122850557290259963615 \cdot 10^{-01}$	$3.70588717878444 \cdot 10^{00}$
41	$6.5637495744757231959699415722023547525400239084778 \cdot 10^{-01}$	$5.93680171871573 \cdot 10^{00}$
44	$9.6886404434506397321859573425794267139455920322171 \cdot 10^{-01}$	$1.01334656625280 \cdot 10^{01}$
53	$-5.6688852568232517859984506645723944339238503996096 \cdot 10^{-01}$	$2.61746665637296 \cdot 10^{01}$
65	$-6.0328072889521477971996798071175156059595671497733 \cdot 10^{-01}$	$7.67818286326206 \cdot 10^{00}$
73	$3.4874711835362408853804154923777452565842552803223 \cdot 10^{-01}$	$2.92507284795068 \cdot 10^{00}$
77	$2.2699132373705254600799448087564660809534768699467 \cdot 10^{-01}$	$3.42067600398534 \cdot 10^{10}$
85	$-7.6894617048676272061865441758881289699552927122551 \cdot 10^{-01}$	$9.90133670369251 \cdot 10^{00}$
1481	$-3.2715595098273932057423414526408419506801164996884 \cdot 10^{00}$	$5.26994449124823 \cdot 10^{00}$
1484	$-1.3432792229759035356265126417855598032166067439989 0 \cdot 10^{01}$	$3.86746474997364 \cdot 10^{01}$
1489	$8.9999999999999999999999999999999999999999999999999 \cdot 10^{00}$	$-3.7 \cdot 10^{-23}$
1496	$1.1199440423162819213593329208218112792285448658029 \cdot 10^{01}$	$2.27616829409607 \cdot 10^{01}$
1501	$-5.8188238119388864901078937905792951783427273426771 \cdot 10^{02}$	$6.06007663972706 \cdot 10^{00}$
4376	$-3.6731327299348159991042234350611468700145535059868 \cdot 10^{02}$	$2.03155740209437 \cdot 10^{01}$
4377	$-5.0062522276143084997015960658866819832113068397294 \cdot 10^{02}$	$2.27150950159608 \cdot 10^{00}$
4393	$6.6000000000000000000000000000000000000000000001468 \cdot 10^{01}$	$5.8 \cdot 10^{-23}$
4396	$-2.3023069110811173762943326075771836710063196221488 \cdot 10^{02}$	$2.00437958330233 \cdot 10^{00}$
4412	$-3.1500483730098996665306117169085504925545562420809 \cdot 10^{02}$	$3.73011222569745 \cdot 10^{01}$

TABLE 5. $E = 37a1$, $P_{-3} \in H_{\frac{1}{2},\rho}$

Δ	$c^+(\Delta)$	$\lvert c^+(\Delta) - [c^+(\Delta)]\rvert$
1489	9	$1.6 \cdot 10^{-72}$
4393	66	$1.5 \cdot 10^{-45}$
5116	-746	$8.5 \cdot 10^{-23}$
5281	153	$8.2 \cdot 10^{-23}$
5560	-1124	$1.2 \cdot 10^{-22}$
5761	-974	$1.1 \cdot 10^{-22}$
6040	-1404	$4.2 \cdot 10^{-23}$
6169	336	$1.1 \cdot 10^{-22}$

TABLE 6. $E = 37a1$, $P_{-3} \in H_{\frac{1}{2},\rho}$. Coefficients are scaled by $\sqrt{3}\, c^-(-3)$.

Δ	$\sqrt{\lvert\Delta\rvert}\, c^-(\Delta)$	$\lvert c^-(\Delta) - [c^-(\Delta)]\rvert$
-4	1	$4.0 \cdot 10^{-84}$
-7	-1	$5.0 \cdot 10^{-84}$
-11	1	$4.5 \cdot 10^{-84}$
-12	-1	$2.0 \cdot 10^{-84}$
-16	-2	$1.1 \cdot 10^{-83}$
-27	-3	$1.3 \cdot 10^{-83}$
-28	3	$1.4 \cdot 10^{-83}$
-36	-2	$1.0 \cdot 10^{-83}$
-40	2	$3.6 \cdot 10^{-85}$
-44	-1	$1.1 \cdot 10^{-83}$

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TABLE 7. $E = 37b1$, $P_{-12} \in H_{\frac{1}{2},\bar{\rho}}$

Δ	$c^+(\Delta)$	$L'(G, \chi_\Delta, 1)$
-3	$1.0267149116920354474451980654263083626994977508118 \cdot 10^{00}$	$1.47929949208 \cdot 10^{00}$
-4	$1.2205364009670316625279102409757685190938519711297 \cdot 10^{00}$	$1.81299789722 \cdot 10^{00}$
-7	$1.6900297463200076214148752932012403965170838158011 \cdot 10^{00}$	$2.11071898018 \cdot 10^{00}$
-11	$5.8849982354849175483779961900586424239744874288522 \cdot 10^{-01}$	$3.65679089534 \cdot 10^{00}$
-40	$1.2669706585839831188366862729215921230412462308976 \cdot 10^{00}$	$4.16362898338 \cdot 10^{00}$
-47	$3.0756790552662277517712909874001702657447701023621 \cdot 10^{00}$	$5.26739088546 \cdot 10^{00}$
-67	$2.1608356105538234382282266707128748893591830597455 \cdot 10^{00}$	$4.98143961845 \cdot 10^{00}$
-71	$-1.5945418432752378367351454423028372659103804842831 \cdot 10^{00}$	$5.33295381308 \cdot 10^{00}$
-83	$2.9631171578917930530100644900469583789690213329975 \cdot 10^{00}$	$7.30522465208 \cdot 10^{00}$
-84	$-3.8773494709413749500399075799371212202544017987791 \cdot 10^{00}$	$1.00026475317 \cdot 10^{01}$
-95	$-2.6554862688645143792016861758519887392731139540185 \cdot 10^{00}$	$5.83606039003 \cdot 10^{00}$
-132	$4.1944733541115532186541330550136737538249181082859 \cdot 10^{00}$	$9.99216716471 \cdot 10^{00}$
-136	$-4.8392675993443437829864850814885823635034770966657 \cdot 10^{00}$	$5.73824076491 \cdot 10^{00}$
-139	$-5.9999999999999999999999999999999999999999999999991 \cdot 10^{00}$	$-8.5 \cdot 10^{-23}$
-151	$-8.3135688179267692046624844818371994826339638923811 \cdot 10^{-01}$	$6.69750855159 \cdot 10^{00}$
-152	$4.3274351625459058613812696410805017025617476195953 \cdot 10^{00}$	$7.95190347996 \cdot 10^{00}$
-811	$-1.4731293182498551151700589944493338505308298027148 \cdot 10^{02}$	$5.32436617837 \cdot 10^{00}$
-815	$1.2194410312093092058885476868302234805268383148338 \cdot 10^{02}$	$4.74925836935 \cdot 10^{00}$
-823	$3.1200000000000000000000000000000000000000000000000 \cdot 10^{02}$	$-1.5 \cdot 10^{-23}$
-824	$-3.22998606660409750567356931348586086493552010570382 \cdot 10^{02}$	$1.75028741141 \cdot 10^{01}$
-835	$-2.4035736526655124690110045874885626910322384359422 \cdot 10^{02}$	$8.64359690730 \cdot 10^{00}$

TABLE 8. $E = 37b1$, $P_{-12} \in H_{\frac{1}{2},\bar{\rho}}$

Δ	$c^+(\Delta)$	$\lvert c^+(\Delta) - [c^+(\Delta)]\rvert$
-139	-6	$1.5 \cdot 10^{-85}$
-823	312	$9.1 \cdot 10^{-80}$
-2051	-26724	$1.0 \cdot 10^{-67}$
-2599	122048	$3.4 \cdot 10^{-63}$
-3223	-472416	$3.2 \cdot 10^{-57}$
-3371	-674712	$7.4 \cdot 10^{-56}$
-5227	5816	$5.5 \cdot 10^{-31}$
-5307	-5192	$4.6 \cdot 10^{-31}$
-6583	-13320	$4.6 \cdot 10^{-31}$
-7892	-79552	$1.2 \cdot 10^{-30}$
-7951	28152	$4.0 \cdot 10^{-31}$
-9112	-224548	$1.6 \cdot 10^{-30}$
-9715	236934	$2.8 \cdot 10^{-32}$
-11444	-1437956	$2.0 \cdot 10^{-33}$
-11651	563716	$7.0 \cdot 10^{-34}$

TABLE 9. $E = 37b1$, $P_{-12} \in H_{\frac{1}{2},\bar{\rho}}$. Coefficients are scaled by $c^-(1)$.

Δ	$\sqrt{\Delta}\, c^-(\Delta)$	$\lvert c^-(\Delta) - [c^-(\Delta)]\rvert$
4	-1	$1.6 \cdot 10^{-85}$
9	0	$3.2 \cdot 10^{-85}$
12	3	$9.6 \cdot 10^{-85}$
16	-2	$1.6 \cdot 10^{-85}$
21	3	$2.7 \cdot 10^{-85}$
25	-1	$1.7 \cdot 10^{-85}$
28	3	$3.9 \cdot 10^{-85}$
33	3	$3.6 \cdot 10^{-85}$
36	0	$3.9 \cdot 10^{-85}$
40	0	$5.6 \cdot 10^{-85}$

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Fachbereich Mathematik, Technische Universität Darmstadt, Schlossgartenstrasse 7, D–64289 Darmstadt, Germany

E-mail address: bruinier@mathematik.tu-darmstadt.de

Fachbereich Mathematik, Technische Universität Darmstadt, Schlossgartenstrasse 7, D–64289 Darmstadt, Germany

E-mail address: stroemberg@mathematik.tu-darmstadt.de